International Journal of Theoretical Physics, Vol. 45, No. 9, September 2006 (© 2006)
DOI: 10.1007/s10773-006-9111-6

Products, Coproducts, and Singular Value Decomposition

Bertfried Fauser [1]

Received September 20, 2005; Accepted February 1, 2006
Published Online: August 17, 2006

Products and coproducts may be recognized as morphisms in a monoidal tensor category of vector spaces. To gain invariant data of these morphisms, we can use singular value decomposition which attaches singular values, i.e. generalized eigenvalues, to these maps. We show, for the case of Grassmannand Clifford products, that twist maps significantly alter these data reducing degeneracies. Since non group like coproducts give rise to non classical behavior of the algebra of functions, makeing them noncommutative, we hope to be able to learn more about such geometries. A remarkabe thechnicallity is that the coproduct for positive singular values of eigenvectors in A yields directly corresponding eigenvectors in $A \otimes A$.

KEY WORDS: products; coproducts; singular value decomposition; noncommutative function algebras.

1. INTRODUCTION

It is well known that function algebras on group manifolds can be recast in a Hopf algebraic setting. The famous Gelfand theorem tells us that every commutative C-*-algebra is dual to the algebra of functions on some topological function space under point wise multiplication. Hence the geometric data can be handled either in the algebraic or in the function theoretic setting.

Since noncommutative C-*-algebras occur naturally. It was an obvious question to ask, which type of geometries are related to the dualized function algebras. However, these function algebras have to be noncommutative. One idea behind this mechanism is the following. Assume there is a point x in a manifold M. We try to find the value of the product of two functions $f, g : M \to \mathbb{C}$ on x

$$(f * g)(x) = f(x)g(x) \tag{1.1}$$

using the *point wise* multiplication of the function values. In other words, the product of two functions is dual to the coproduct on the points of the manifold.

[1] Max Planck Institut für Mathematik, Inselstrasse 22-26, D-04103 Leipzig, Germany; e-mail: Bertfried.Fauser@uni-konstanz.de.

Symbolically

$$
\begin{aligned}
(f * g)(x) = (m(f \otimes g))(x) &= (f \otimes g)(m^*(x)) = (f \otimes g)(x \otimes x) \\
&= f(x)g(x).
\end{aligned}
\tag{1.2}
$$

Here we had to assume that the coproduct m^* is group like, i.e. $m^*(x) = x \otimes x$, and that the evaluation map $\mathsf{eval}(f \otimes x) = f(x)$ is generalized canonically as a crossed map to $\mathsf{eval}_{(V^*)^{\otimes 2} \otimes V^{\otimes 2}}((f \otimes g) \otimes (x \otimes y)) = f(x)g(y)$. While this mechanism seems to be natural, since we have used it already in high school, it can readily be generalized to the case where one demands that the coproduct is non-group like. In Chryssomalakos, (1998) one may look up a detailed description of that point of view, and what may change in the underlying geometry.

A second source of noncommutativity is related to twist maps and 'quantization' (Fauser, 1996, 2002; Hirshfeld and Henselder, 2002). Such twist maps can be subsumarized under the name of *cliffordization*. An alternative name would be comodule algebra map. We prefer the former in combinatorially intended settings. The twisted product of two morphisms is given as

$$
f *_\chi g = \sum_{(f)(g)} \chi\left(f_{(1)}, g_{(1)}\right) f_{(2)} \times g_{(2)} \tag{1.3}
$$

where we have used Sweedler indices $m^*(f) = f_{(1)} \otimes f_{(2)}$ to denote efficiently the sum of tensors apearing in the coproduct, see (Sweedler, 1969). It is easily seen, that this leads in general equally well to a non-commutative function algebra. This particular twist was discussed also in Oziewicz, (2001), where for the graphical representation the term *Rota sausage* was coined.

An easily tractable and fruitfull model of such a deformation is the transition from the Grassmann Hopf algebra (or symmetric Hopf algebra) to the Clifford comodule algebra (or Weyl comodule algebra) as described in detail in Rota and Stein, (1994). Hence it might be useful to skip all further complications and to investigate the product and coproduct structure in such algebras. A natural way to study such deformations is using cohomological methods (Sweedler, 1968). This led to amazing insights into the structure of quantum field theories (Brouder *et al.*, 2003) and symmetric functions (Fauser and Jarvis, 2004). While this method produced even computational tools and is suited for super algebras etc, we want to take in this paper another route.

Any product in an algebra A is a linear morphisms $m : A \otimes A \to A$. Seen in the category of modules, its just a module morphism from $B = A \otimes A$ to A. Hence, assuming finite dimensionality for the sake of simplicity, and introducing bases, we get a rectangular representation matrix for a product morphism, i.e. the multiplication table, characterizing the morphism. Let $\{a_i\}$ be a basis of A

and $\{b_I \cong (a_i \otimes a_j)\}$ a basis of $B = A \otimes A$, we get

$$m(a_i \otimes a_j) = m(b_I) = \sum_k m_I^k a_k = \sum_k m_{ij}^k a_k . \tag{1.4}$$

The change of perspective of formally reducing a tensor of degree three to a tensor of degree two has to be payed for by dealing with rectangular matrices. The same holds true for coproducts, where we easily see, that products and coproducts of a Hopf algebra obeys representations[2] as $n \times m$ and $m \times n$ matrices, which allows to concatenate them. Notationally, we will use lower case indices for elements in A and upper case indices for elements in $B = A \otimes A$. Let α be the isomorphism which encodes and decodes the two index sets, hence

$$\alpha(I) = [i, j] \quad \alpha^{-1}([i, j]) = I. \tag{1.5}$$

A matrix representation of α is a tensor of degree three $\alpha_{ij}^I \in B \otimes (A^* \otimes A^*)$. Having done this, we can apply the techniques from ordinary linear algebra among them singular value decomposition to characterize product and coproduct maps. We will see in the course of this work that this information is more subtle and detailed then the above mentioned cohomological classification and therefore opens up new theoretical insight. Moreover, it is well known from singular value theory, that the large singular values characterize a rectangular map reasonably well. Hence we might hope to expand products and coproducts using only a few large singular values, dropping small ones without great loss of information. In this way, we may hope to develop a method, which will allow to replace complicated product and coproduct structures in a coherent way, maintaining the Hopf algebra structure, by a much simpler and well adapted product coproduct pair. Ultimately we hope to get geometrical insights via this approach as well.

2. HOPF ALGEBRA STRUCTURE

While this work could be formulated with quite small stock of mathematics, we want to introduce some Hopf algebra notions to be able to set our results up in that framework. However we do not to assume much knowledge about Hopf algebras providing roughly the axioms here. Some of the formulas are needed for reference issues later. References for Hopf algebra theory may be (Sweedler, 1969; Abe, 1980; Kassel, 1995; Majid, 1995). The reader interested in the matrix versions might like to proceed to Section 4 and come back to this and the next section when necessary.

Let A be an associative, unital $\mathbb{k}$-algebra. We denote the underlying $\mathbb{k}$-module of A by abuse of notation also with A. The product map is denoted

[2] We assume A to be of dimension n, the dimension of B is then $m = n \times n$. However, our arguments run through without this speacialisation for arbitrary spaces A and B.

$m : A \otimes_{\mathbb{k}} A \to A$ and is a $\mathbb{k}$-linear map of modules in the monoidal category of $\mathbb{k}$-modules $\mathsf{mon}_{\mathbb{k}}$. The unit is $\eta : \mathbb{k} \to A$. The monoid forms a symmetric tensor category with respect to the twist map $\mathsf{sw} : A \otimes B \to B \otimes A$. Note that the switch map has to be universal (natural), hence one has to impose a coherence law which in this case is given by the braid equation. For our purpose important is the fact, that the switch map is represented as a permutation matrix P under the α isomorphism

$$\alpha \circ \mathsf{sw}(A \otimes A) = \mathsf{P} \circ \alpha(A \otimes A) \; = \; \mathsf{P}\alpha(B). \tag{2.6}$$

Let C be an coassociative, counital $\mathbb{k}$-coalgebra. We denote the underlying $\mathbb{k}$-comodule of C by abuse of notation also with C. The coproduct map is denoted $\delta : C \to C \otimes C$ if group like ($\delta(x) = x \otimes x$) and $\Delta : C \to C \otimes C$ if not group like ($\Delta(x) = x_{(1)} \otimes x_{(2)}$ implicite sum). The counit is denoted $\epsilon : C \to \mathbb{k}$. δ and Δ are morphisms in $\mathsf{mon}_{\mathbb{k}}$. We adopt the Brouder-Schmitt convention (Brouder and Schmitt, 2002), denoting the Sweedler indices of the coproduct of $\delta(x) = x_{[1]} \otimes x_{[2]}$ and $\Delta(x) = x_{(1)} \otimes x_{(2)}$ using different bracings.

An algebra A (coalgebra C) is called augmented, if it has an (co)augmentation morphisms, a counit $\epsilon : A \to \mathbb{k}$ (an unit $\eta : \mathbb{k} \to C$). An (co)augmented (co)algebra is called connected, if the (co)augmentation as an (co)algebra map satisfies

$$\epsilon \circ m \; = \; m_{\mathbb{k}} \circ (\epsilon \otimes \epsilon) \quad \Delta \circ \eta \; = \; (\eta \otimes \eta) \circ \delta^{\mathbb{k}} \tag{2.7}$$

It is known that twists of connected (co)algebras lead in general to nonconnected (co)algebras even if the twist is cohomologically trivial, i.e. induced via a 2-coboundary (Brouder *et al.*, 2003; Fauser and Jarvis, 2004). Such algebras were coined 'interacting' in Fauser, (2002).

A bialgebra[3] is a module B carrying an algebra structure m and a coalgebra structure Δ such that the compatibility law

$$\Delta \circ m = (m \otimes m)(\mathrm{Id} \otimes \mathsf{sw} \otimes \mathrm{Id})(\Delta \otimes \Delta) \tag{2.8}$$

holds. This states that m, (Δ) is a coalgebra (algebra) homomorphism. This compatibility law allows actual computations since it embodies the germ of Laplace expansions together with the dual Hopf algebra.

A Hopf algebra H is a bialgebra where an antipode $\mathsf{S} : H \to H$ exists, fulfilling

$$\mathsf{S}\big(x_{(1)}\big)x_{(2)} = \eta \circ \epsilon(x) \; = \; x_{(1)}\mathsf{S}\big(x_{(2)}\big). \tag{2.9}$$

It is possible to start with the convolution demanding the existence of an antipode. It was proved by Oziewicz that any antipodal convolution has a crossing which fulfills Eq. (2.8). However, the crossing needs not to be the switch and even

[3] We use the common letter B for bialgebra, not to be confused with the B intoduced above.

not to be be a braid. Such algebras were denoted *Hopf gebras*, see (Fauser, 2002) for details. The methods developed are in principle aplicable to this situation too.

3. GRASSMANN HOPF ALGEBRA AND TWISTS

To simplify our discussion, we will study Grassmann Hopf algebras $\Lambda(V) = (V^{\wedge}, \wedge)$, which are computationally manageable and provide nevertheless an archetypical example. Let V be a finite dimensional vector space, the exterior powers of V are denoted as $V^{\wedge^r}$, which sum up to a graded space $V^{\wedge} = \sum V^{\wedge^r}$. The product is given by the exterior product $\wedge$ (wedge product) and the coproduct is given recursively by

$$\begin{aligned} \Delta(v) &= v \otimes \mathrm{Id} + \mathrm{Id} \otimes v \quad v \text{ in } V \\ \Delta(A \wedge B) &= \sum \pm A_{(1)} \wedge B_{(1)} \otimes A_{(2)} \wedge B_{(2)} \;=\; \Delta(A)\Delta(B) \end{aligned} \tag{3.10}$$

where the sign is given by the alternating character of the symmetric group yielding the graded switch $\mathsf{sw}(V^{\wedge^r} \otimes V^{\wedge^s}) = (-1)^{rs} V^{\wedge^s} \otimes V^{\wedge^r}$ for the crossed products and extended by linearity.

The antipode is given as $\mathsf{S}(V^{\wedge^s}) = (-1)^s\, V^{\wedge^s}$ and the counit is given as $\epsilon(\mathrm{Id}) = 1$, $\epsilon(V^{\wedge^r}) = 0$ for all $r > 0$.

Let $\{e_i\}$ be a basis of V, a basis for elements of $V^{\wedge^r}$ is given by $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r}\}$ where $i_1 < i_2 < \cdots < i_r$.

We can now introduce a new product, called cliffordization or circle product, using a general bilinear form $B^{\wedge}$ on $V^{\wedge} \otimes V^{\wedge}$ induced from a bilinear form $B : V \otimes V \to \Bbbk$ as

$$x \circ y = \sum_{(x),(y)} \pm B^{\wedge}\big(x_{(1)}, y_{(1)}\big)\, x_{(2)} \wedge y_{(2)}\,. \tag{3.11}$$

The bilinear form is evaluated by Laplace expansion

$$\begin{aligned} B(\Bbbk, V) \;=\; 0 &= B(V, \Bbbk) \quad B : V \otimes V \to \Bbbk \\ B^{\wedge}(x \wedge y, z) &= \sum_{(z)} \pm B^{\wedge}\big(x, z_{(1)}\big) B^{\wedge}\big(y, z_{(2)}\big) \\ B^{\wedge}(x, y \wedge z) &= \sum_{(x)} \pm B^{\wedge}\big(x_{(1)}, y\big) B^{\wedge}\big(x_{(2)}, z\big). \end{aligned} \tag{3.12}$$

Since $B^{\wedge}$ is Laplace, it is a 2-cocycle and the circle product is associative. We know from (Fauser, 2001b; Brouder *et al.*, 2003) that we can distinguish two cases of such twists. If $B^{\wedge}$ is antisymmetric, then the twisted algebra is isomorphic to the original algebra. $B^{\wedge}$ is a 2-coboundary in this case. However, the original grading remains only a filtration but can be newly established with respect to the new product. That means we find in this case an isomorphism

$$\begin{aligned}\Phi\ &:\ V^{\wedge} \to V^{\circ}\\ \Phi(V^{\wedge^r}) &= V^{\circ^r} \oplus V^{\circ^{(r-2)}} \oplus \dots \\ \text{and}\qquad V^{\circ} &= \sum_r V^{\circ^r}.\end{aligned} \tag{3.13}$$

This is the famous Wick expansion of quantum field theory (Fauser, 2001b). If B is symmetric, then the map is no longer an algebra isomorphism. The resulting algebra is the Clifford algebra of the quadratic space (V, Q), $Q(x) = B(x, x)$. Both cases can be combined to come up with an arbitrary bilinear form. Our further objective is to implement new tools to study these two cases of twist deformation.

4. SINGULAR VALUE DECOMPOSITION

To be able to develop our new viewpoint, we need to address product and coproduct maps as morphisms in mon. Hence we introduce a linearly ordered $\{e\}$-basis in $V^{\wedge}$ of dimension $2^{\dim V}$. We consider from now on the whole graded space $V^{\wedge}$ and this basis is linearly indexed. If we focus on the generating space V, we will explicitely mention this. Using this convention we obtain the maps

$$\begin{aligned} m(e_i \otimes e_j) &= \sum_k m_{ij}^k e_k \;=\; \sum_k m_I^k e_k \\ \Delta(e_i) &= \sum_{(e_i)} \pm \Delta_i^{kj} e_k \otimes e_j \;=\; \sum_{(e_K)} \pm \Delta_i^K e_K \end{aligned} \tag{4.14}$$

where $\{e_K\}$ is a linearly ordered basis of $V^{\wedge} \otimes V^{\wedge}$ and α defined in Eq. (1.5) is the encoding isomorphism $\alpha^{-1}(e_i \otimes e_j) = e_K$. It is obvious that m_I^k is a $4^n \times 2^n$-tensor, while Δ_k^I is an $2^n \times 4^n$-tensor.

To be able to derive invariant informations, like eigenvalues, we need to associate quadratic matrices to m and Δ. We will concentrate on m, since Δ is treated analogously. Let m^{T} denote the *transposed matrix* of m, i.e. rows and columns interchanged. m^{T} is a $2^n \times 4^n$-matrix. To be precise, m^{T} is the coproduct of the dual Hopf algebra H^*. To see this, let $\{f^i\}$ be the linearly ordered canonical dual basis of the $\{e\}$ basis. We have

$$\begin{aligned} \mathsf{eval}(f^i \otimes e_j) &= f^i(e_j) = \delta_j^i \\ \mathsf{eval}(f^i \wedge' f^j \otimes e_k \wedge e_l) &= \frac{1}{4}\mathsf{eval}((f^i \otimes f^j - f^j \otimes f^i) \otimes (e_k \otimes e_l - e_l \otimes e_k)) \\ &= \frac{1}{2}\left(\delta_k^i \delta_l^j - \delta_l^i \delta_k^j\right) \end{aligned} \tag{4.15}$$

etc. It is now possible to combine the morphisms m and m^{T} in two ways

$$A = m \circ m^{\mathrm{T}} \quad B = m^{\mathrm{T}} \circ m\,, \tag{4.16}$$

where A is a $2^n \times 2^n$-matrix and B is a $4^n \times 4^n$-matrix. Both matrices are symmetric by construction and can be diagonalized by an unitary (orthogonal) matrix ($U : V^{\wedge} \to V^{\wedge}$, and $V : (V^{\wedge} \otimes V^{\wedge}) \to (V^{\wedge} \otimes V^{\wedge})$)

$$\begin{aligned} D_A &= U A U^{\mathrm{T}} \quad D_B = V B V^{\mathrm{T}} \\ U\,U^{\mathrm{T}} &= \mathrm{Id}_V \quad V\,V^{\mathrm{T}} = \mathrm{Id}_{V \otimes V}. \end{aligned} \tag{4.17}$$

Since $A^{\mathrm{T}} = A$ and $B^{\mathrm{T}} = B$ are non-negative matrices, we can compute the square root of D_A and D_B using functional calculus. We denote by $D_A^{\frac{1}{2}}$ the $n \times m$ respectively $(D_A^{\frac{1}{2}})^{\mathrm{T}}$ the $m \times n$ matrices which recombine to $D_A = D_A^{\frac{1}{2}}(D_A^{\frac{1}{2}})^{\mathrm{T}}$ and D_B analogously. For ease of notation we drop the transposition $^{\mathrm{T}}$ since the shape of $D_A^{\frac{1}{2}}$ is obviouis from the context. This allows us to write

$$\begin{aligned} A &= U^{\mathrm{T}} D_A^{\frac{1}{2}} D_A^{\frac{1}{2}} U = U^{\mathrm{T}} D_A^{\frac{1}{2}} V\, V^{\mathrm{T}} D_A^{\frac{1}{2}} U \;=\; m \circ m^{\mathrm{T}} \\ B &= V^{\mathrm{T}} D_B^{\frac{1}{2}} U\, U^{\mathrm{T}} D_B^{\frac{1}{2}} V \;=\; m^{\mathrm{T}} \circ m. \end{aligned} \tag{4.18}$$

Therefore one concludes that $D_A \oplus 0_{\mathsf{dim\ ker}(m)} = D_B$ (0_k the $k \times k$ zero matrix) and especially that the *sets* of positive eigenvalues of A and B are identical. The eigenvalues of D_A or D_B are called singular values, they are nonnegative by construction. One has to be careful during the identification of the two maps since they agree only up to isomorphism (a permutation of the singular values). However this fact allows imediately to state without further proof the following theorem, which was proved originally by Oziewicz via laborious computations:

Theorem 4.1. (Oziewicz, 2001, pp. 184–185) *The operators $m \circ \Delta$ and $\Delta \circ m$ fulfill the same minimal polynomial and differ only in the dimension of their kernels.*

This is quite important, since it is also a statement about the right-hand-side of Eq. (2.8), a fundamental axiom of bi- and Hopf algebras!

Let now $\{u_i\}$ be the set of column vectors of U and $\{v_I\}$ be the set of column vectors of V and let $\{d_i\}$ be the set of positive singular values of D_A or D_B. It is now possible to relate the two sets of vectors via

$$\begin{aligned} m\, v_I &= \pm\big(d_A^{\frac{1}{2}}\big)_{\alpha^{-1}([i,1])}\, u_{\alpha^{-1}([i,1])} \;\cong\; \pm\big(d_A^{\frac{1}{2}}\big)_i\, u_i \\ m^{\mathrm{T}} u_i \;&\cong\; m^{\mathrm{T}} u_{\alpha(I)} = \pm\big(d_B^{\frac{1}{2}}\big)_I\, v_I. \end{aligned} \tag{4.19}$$

Using that particular isomorphism α which relates the index sets $\{I\}$ and $\{i\}$ in such a way that $\alpha^{-1}(I) = [i, 1]$ picks eigenvectors to the same singular value $d_i^{\frac{1}{2}}$ One can then come up with a *spectral decomposition* of the product and coproduct maps. Our choice of the signs in the square roots fixes the maps completely. Note that eigenvectors are assumed to be nonzero, orthogonal and normalized. However, from $v_i \cdot v_i = 1$ we can fix v_i only up to sign. We may hence choose positive signs, finding

$$\begin{aligned} m &= \sum_i u_i \left(d_A^{\frac{1}{2}}\right)_i v^{\mathrm{T}}_{\alpha^{-1}([i,1])} \\ m^{\mathrm{T}} &= \sum_i v_{\alpha^{-1}([i,1])} \left(d_A^{\frac{1}{2}}\right)_i u_i^{\mathrm{T}}. \end{aligned} \tag{4.20}$$

where the sum is over all positive singular values.

5. SINGULAR VALUE DECOMPOSITION FOR GRASSMANN AND CLIFFORD ALGEBRAS

5.1. Grassmann Case

We proceed to calculate explicitly the singular values for Grassmann and Clifford algebra products and coproducts of course. We start with the Grassmann case and compute the d_i for the composition $A = m \circ \Delta$. Therefore we note that the coproduct of a basis element $e_{i_1} \wedge \cdots \wedge e_{i_r}$ is given by all (p, q) -shuffles of the indices $(i_1, \ldots, i_r)$, where $p + q = r$. Wedging each of these terms back, one obtains the original basis element. Hence we find

$$m \circ \Delta(e_{i_1} \wedge \cdots \wedge e_{i_r}) = \#\text{of } (p, q)\text{-shuffles } \cdot (e_{i_1} \wedge \cdots \wedge e_{i_r}). \tag{5.21}$$

To compute the number of (p, q) -shuffles with $p + q = r$, we need to select zero, one, two, etc elements out or r elements, getting r choose p such sequences. Summing up, we get 2^r terms. If we introduce the grade operator ∂ as

$$\partial : \oplus V^{\wedge^r} \to \mathbb{N} \quad \partial V^{\wedge^r} = r \tag{5.22}$$

we can write our result as

Theorem 5.2. (Oziewicz, 1997) *The operator* $A = m^\wedge \circ \Delta^\wedge$ *acts as the linear operator* 2^∂ *on* $V^\wedge$.

This is a well known result, but we can generalize this in the following way.

Theorem 5.3. *The operator* $A^{(r)} = (m^\wedge)^{r-1} \circ (\Delta^\wedge)^{r-1}$ *acts as linear operator* r^∂ *on* $V^\wedge$ *(* $A = A^{(2)}$ *,* $A^{(1)} = Id$ *).*

Proof: We need to count the number of $(p_1, \ldots, p_r)$-shuffles with $\sum p_i = \dim V$, related to multinomial coefficients, while we had to count binomial coefficients in the preceding theorem. □

Note that these operators are homogeneous with respect to the grade and commute with the grade operator ∂. Hence they are constant on each space $V^{\wedge^r}$. Hence we cannot drop smaller eigenvalues since all of them are equal on all homogeneous elements. However, the higher grade elements have larger singular values. A map $F : V^\wedge \to V^\wedge$ can hence be considered to have more weight on the higher grade subspaces.

Knowing the singular values, we can easily write down the minimal polynomial of the operators A and $A^{(r)}$

$$\prod_{i=0}^{\dim V} (A - 2^i) = 0 \qquad \prod_{i=0}^{\dim V} \left(A^{(r)} - r^i\right) = 0 \tag{5.23}$$

The geometric degeneracies of the eigenspaces are given by binomial and multinomial coefficients and we can infer the characteristic polynomials too, e.g.

$$\begin{aligned} \prod_{i=0}^{\dim V} (A - 2^i)^{\binom{\dim V}{i}} &= 0 \\ B^{(4^{\dim V} - 2^{\dim V})} \prod_{i=0}^{\dim V} (B - 2^i)^{\binom{\dim V}{i}} &= 0. \end{aligned} \tag{5.24}$$

The grade operator applied directly to the index set returns simply the cardinality of the index set $|\{i_1, \ldots, i_r\}| = r$. The Grassmann product and coproduct maps have therefore the spectral decomposition

$$\begin{aligned} m &= \sum_{i=1}^{\dim V} u_i \; 2^{\frac{1}{2}\,|\alpha^{-1}([i,1])|} \; v^{\mathrm{T}}_{\alpha^{-1}([i,1])} \\ m^{\mathrm{T}} = \Delta &= \sum_{i=1}^{\dim V} v_{\alpha^{-1}([i,1])} \; 2^{\frac{1}{2}\,|i|} \; u_i^{\mathrm{T}}, \end{aligned} \tag{5.25}$$

where the sum is over all nonzero singular values and α is the particular index isomorphism which guarantees that u_i and $v_{\alpha^{-1}([i,1])}$ belong to the same singular value.

5.2. Clifford Case

The Clifford case is much more involved. We can distinguish three cases. Either deform the product, the coproduct or both. Since we use ordinary transposition to obtain $m^{\mathrm{T}} = \Delta$, hence identifying the Hopf algebras H and H^*, we cannot do this independently unless we allow nontrivial dual isomorphisms. In this case the dual basis is given by $f^i(e_j) = h^i_j$ where h^i_j is a $GL(n)$ element. While this may be of importance in geometry and physics, see Fauser and Stoss (2004), we will not include here this complication. We will use product deformations and the coproduct is deformed by demanding an Euclidean duality isomorphism, i.e. δ^i_j.

However, we will allow deformations by symmetric or nonsymmetric bilinear forms. We will postpone the general case to the computer algebra experiment and concentrate here on the following situation: Let $g : V \otimes V \to \mathbb{k}$ be a symmetric non degenerate bilinear form. Let $\Delta(g) = g_{(1)} \otimes g_{(2)}$. We define the twisted (Clifford or circle) product and coproduct maps as

$$
\begin{aligned}
m_g(x \otimes y) = x \circ_g y &= \sum_{(x),(y)} (-1)^{\partial x_{(2)} \partial y_{(1)}} g^\wedge\big(x_{(1)}, y_{(1)}\big) x_{(2)} \wedge y_{(2)} \\
\Delta_{g'}(x) &= \sum_{(x)} (-1)^{\partial g'_{(2)} \partial x_{(1)}} g'_{(1)} \wedge x_{(1)} \otimes g'_{(2)} \wedge x_{(2)}
\end{aligned} \tag{5.26}
$$

The coproduct with respect to the metric g can be written as

$$
\begin{aligned}
\Delta_g(x) = \mathrm{Id} \otimes \mathrm{Id} &+ \sum_{i,j} g_{ij} x_i \otimes x_j \\
&+ \sum_{i<j,k<l} \frac{1}{2!}(g_{ik}g_{jl} - g_{il}g_{jk}) x_i \wedge x_j \otimes x_k \wedge x_l + \cdots
\end{aligned} \tag{5.27}
$$

where the decomposable element x is given as a monomial in the x_i and the expression is extended by linearity to $V^\wedge$, see Fauser, (2002). From the preceding two expressions we deduce, that the coproduct Δ obtained by transposition of the multiplication table m^k_{ij} is given by the deformation w.r.t. the numerically identical cometric g', i.e. we have $g \equiv g'$.

Example 5.4. Let $\dim V = 1$ and introduce the metric $g(e_1, e_1) = a$. We use the basis $\{\mathrm{Id} = e_0, e_1\}$ for $V^\wedge$ and $\{\mathrm{Id} \otimes \mathrm{Id}, e_1 \otimes \mathrm{Id}, \mathrm{Id} \otimes e_1, e_1, \otimes e_1\}$ for $V^\wedge \otimes V^\wedge$, with shorthand $\{e_{0,0}, e_{1,0}, e_{0,1}, e_{1,1}\}$. We find the multiplication table and the section coefficients (comultiplication table)

$$m_g \cong \begin{array}{c|cccc} m_g & e_{0,0} & e_{1,0} & e_{0,1} & e_{1,1} \\ \hline e_0 & 1 & 0 & 0 & a \\ e_1 & 0 & 1 & 1 & 0 \end{array} \qquad m_g^{\mathrm{T}} \cong \begin{array}{c|cc} m_g^{\mathrm{T}} & e_0 & e_1 \\ \hline e_{0,0} & 1 & 0 \\ e_{1,0} & 0 & 1 \\ e_{0,1} & 0 & 1 \\ e_{1,1} & a & 0 \end{array} \tag{5.28}$$

The matrices $A = m_g \circ m_g^{\mathrm{T}}$ and $B = m_g^{\mathrm{T}} \circ m_g$ read then

$$A \cong \begin{bmatrix} 1+a^2 & 0 \\ 0 & 2 \end{bmatrix}$$
$$B \cong \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ a & 0 & 0 & a^2 \end{bmatrix} \tag{5.29}$$

The eigenvalues are hence $1+a^2$, 2 leading to the singular values $\sqrt{1+a^2}$, $\sqrt{2}$. The matrix A is already diagonal, showing that e_0, e_1 are orthonormalized eigenvectors $\{u_i\}$. However, we need to orthogonalize B. We can arrange the new basis $\{v_i\}$ as

$$\begin{aligned} \lambda = (1+a^2): \quad & v_1 = \frac{1}{\sqrt{1+a^2}}(\mathrm{Id} \otimes \mathrm{Id} + a\, e_1 \otimes e_1) \\ \lambda = 2: \quad & v_2 = \frac{1}{\sqrt{2}}(e_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes e_1) \\ \lambda = 0: \quad & v_3 = \frac{1}{\sqrt{2}}(e_1 \otimes \mathrm{Id} - \mathrm{Id} \otimes e_1) \\ \lambda = 0: \quad & v_4 = \frac{1}{\sqrt{1+a^2}}(a\, \mathrm{Id} \otimes \mathrm{Id} - e_1 \otimes e_1) \end{aligned} \tag{5.30}$$

Note, that the product map acting on the v_i yields the square root of the singular values times the column (eigen)vectors u_i. Especially $m(v_3) = 0$ and $m(v_4) = 0$, showing that $\mathsf{ker}(m) \cong \mathsf{lin\text{-}hull}(v_3, v_4)$. The product and coproduct spectral decompositions are given as

$$\begin{aligned} m(x \otimes y) &= \sum_{i=1}^{2} u_i d_i^{\frac{1}{2}} v_i^{\mathrm{T}}(x \otimes y) \\ &= \mathrm{Id}\sqrt{1+a^2}\frac{1}{\sqrt{1+a^2}}(\mathrm{Id}(x) \otimes \mathrm{Id}(y) + a\, e_1(x) \otimes e_1(y)) \\ &\quad + e_1\sqrt{2}\frac{1}{\sqrt{2}}(e_1(x) \otimes \mathrm{Id}(y) + \mathrm{Id}(x) \otimes e_1(y)) \end{aligned}$$

$$= g(\mathrm{Id}, x)g(\mathrm{Id}, y) + a\, g(e_1, x)g(e_1, y) + e_1(g(e_1, x)g(\mathrm{Id}, y) + g(\mathrm{Id}, x)g(e_1, y)) \tag{5.31}$$

$$m^{\mathrm{T}}(x) = \sum_{i=1}^{2} v_i d_i^{\frac{1}{2}} u_i^{\mathrm{T}}(x) = (\mathrm{Id} \otimes \mathrm{Id} + a\, e_1 \otimes e_1)g(\mathrm{Id}, x) + (e_1 \otimes \mathrm{Id} - \mathrm{Id} \otimes e_1)g(e_1, x) \tag{5.32}$$

Setting $a = 0$ returns the Grassmann case. The particular choice $a = \pm i$, the complex number unit, increases the degeneracy and one has a three dimensional null space. Note that one eigenvalue is equal to $\dim V^{\wedge} = 2$, but that the other one depends in general on a.

This illuminates the following

Theorem 5.5. (Oziewicz, 2001) *If m_g is twisted by a symmetric nondegenerate bilinear form g and $\Delta_{g^{-1}}$ is deformed by g^{-1} then the operator $A = m_g \circ \Delta_{g^{-1}}$ acts as the multiplication by* $\dim V^{\wedge}$.

This theorem can readily be generalized.

Theorem 5.6. *If m_g is twisted by a symmetric nondegenerate bilinear form g and $\Delta_{g^{-1}}$ is twisted by g^{-1}, then the operators $A^{(r)} = m_g^{r-1} \circ \Delta_{g^{-1}}^{r-1}$ acts as multiplication operators $(dim V^{\wedge})^{r-1}$, in particular $A^{(2)} \cong \dim V^{\wedge}$.*

Proof: A trivial iteration of the preceding theorem. □

In particular our outcome shows directly that the condition that the deformations are mutually related via the inverse metrics is necessary.

Theorem 5.7. (Oziewicz, 1997) *If the cliffordization is performed w.r.t. a (symmetric) metric g and the coproduct is deformed w.r.t. the cometric g^{-1}, such that $gg^{-1} = Id$, then the convolution has no antipode.*[4]

This result renders the codeformation w.r.t. the inverse to be a particular singular and unuseful situation if a pseudoinverse (antipode) is needed. Especially in physics a pseudoinverse is desirable in most cases. A way out of this problem was investigated in Fauser and Oziewicz, (2001).

[4] The "moves" decribed by the two preseeding theorems are known as *Frobenius conditions* and play some role in cobordism theory, and 2d topological quantum field theory.

In our case, since we had demanded that $\Delta = m^{\mathrm{T}}$, we obtain the singular case for symmetric metrics fulfilling $g^2 = \mathrm{Id}$.[5] All these matrices are in the orbit of diagonal matrices with diagonal entries ± 1. Due to Sylvester's theorem, all $GL(n, \Bbbk)$ matrices fall into an orbit of such an element, as long as the ground field $\Bbbk$ is of characteristic 0. In particular, $GL(n, \mathbb{C})$ has one such orbit while $GL(n, \mathbb{R})$ hast $n + 1$ such orbits, characterized using the signature. This is related to the Brauer-Wall group of quadratic forms, see (Hahn, 1994). Hence after normalizing $g(e_1, e_1) = a = 1$ we find in our above example Oziewicz's result. However, spin groups or special orthogonal groups, as symplectic groups do not allow such a rescaling. For a brief relation of this outcome to group branching laws see Section 7.

Let us deviate a little bit from Clifford topics and consider the group like coproduct $\delta(x) = x \otimes x$ for all x. One can show, that the pair of morphisms $m^\wedge, \delta$ still fulfills the axiom (2.8), but that in general for this and twisted such products no antipode exists. Dualizing this time the comultiplication, results in a product map $\delta^{\mathrm{T}} = m^{\mathrm{B}}$. This product turns V into a Boolean algebra (all elements are idempotent)

$$m^B(x \otimes y) = \begin{cases} x & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \tag{5.33}$$

The matrix $A = m^{\mathrm{B}} \circ \delta$ is the unit matrix in $\dim V$ dimensions and $B = \delta \circ m^{\mathrm{B}}$ is a diagonal matrix with $\dim V$ ones and zeros otherwise. A twist deformation in this case transforms the elements from being idempotent to being almost idempotent, hence an uninteresting map. However, disregarding the transposition as being Euclidean, we can combine $m^\wedge \circ \delta$, which is related to inner products of group representations and nontrivial. Note that for group like situations we obtain full degeneracy of singular values. Hence the classical geometric case is characterized by total degeneracy of the product and coproduct maps.

Now, let us assume that we have a symmetric bilinear form g. It is possible to diagonalize this form in the space V, we can deduce then $g^\wedge$ and obtain for the matrix $A = m_g^{\mathrm{T}} \circ m_g$ the diagonal representation

$$\begin{aligned} g &: V \otimes V \to \Bbbk \quad g = \mathrm{diag}(l_1, \ldots, l_n) \\ g^\wedge &: V^\wedge \otimes V^\wedge \to \Bbbk \\ g^\wedge &= \mathrm{diag}\Big(L_0, 2^{\binom{n}{1}} L_i^{\binom{n}{1}}, 2^{\binom{n}{2}} L_{ij}^{\binom{n}{2}}, \ldots, 2^{\binom{n}{n-1}} L_{i_1,\ldots,i_{n-1}}^{\binom{n}{n-1}}, 2^{\binom{n}{n}}\Big) \end{aligned} \tag{5.34}$$

where we have split off the Grassmann eigenvalues $2^{\binom{n}{m}}$, and the metric dependent parts $L_{i_1\ldots}$. Superscripts of the $L_{i\ldots}$ denote the 'degeneracy' of eigenvalues of the

[5] For those readers curious about domains and codomains of maps, this means that we establish an explicite isomorphism between V and V^* identifying e_i with e^i for all i. This renders the Hopf algebra to be self dual.

same type but different index structure. The subscripts denote which indices are *missing* in the total index set $\{1,\ldots,\mathrm{dim}V\}$. The $L_{i_1,\ldots,i_r}$ read as follows, where the sums range over in $\{1,\ldots,n\}$ omitting $\{i_1,\ldots,i_r\}$ which index the basis of $V^\wedge$. It is clear that there are $\binom{n}{r}$ such sets which explains the 'degeneracy'. Hence we find

$$\begin{aligned}
L_0 &= 1+\sum_{i_1} l_{i_1}^2+\sum_{i_1<i_2} l_{i_1}^2 l_{i_2}^2+\sum_{i_1<i_2<i_3} l_{i_1}^2 l_{i_2}^2 l_{i_3}^2+\cdots+\sum_{i_1<\cdots<i_{n-1}} l_{i_1}^2\ldots l_{i_{n-1}}^2\\
L_i &= 1+\sum_{i_1} l_{i_1}^2+\sum_{i_1<i_2} l_{i_1}^2 l_{i_2}^2+\sum_{i_1<i_2<i_3} l_{i_1}^2 l_{i_2}^2 l_{i_3}^2+\cdots+\sum_{i_1<\cdots<i_{n-2}} l_{i_1}^2\ldots l_{i_{n-2}}^2\\
L_{ij} &= 1+\sum_{i_1} l_{i_1}^2+\sum_{i_1<i_2} l_{i_1}^2 l_{i_2}^2+\sum_{i_1<i_2<i_3} l_{i_1}^2 l_{i_2}^2 l_{i_3}^2+\cdots+\sum_{i_1<\cdots<i_{n-3}} l_{i_1}^2\ldots l_{i_{n-3}}^2\\
L_{ijk} &= 1+\sum_{i_1} l_{i_1}^2+\sum_{i_1<i_2} l_{i_1}^2 l_{i_2}^2+\sum_{i_1<i_2<i_3} l_{i_1}^2 l_{i_2}^2 l_{i_3}^2+\cdots+\sum_{i_1<\cdots<i_{n-4}} l_{i_1}^2\ldots l_{i_{n-4}}^2\\
&\ldots\\
L_{i_1,\ldots,i_r} &= \sum_{s=0}^{r}\sum_{i_1<\cdots<i_s} l_{i_1}^2\ldots l_{i_{n-s-1}}^2\\
&\ldots\\
L_{i_1,\ldots,i_n} &= 1
\end{aligned}\tag{5.35}$$

The sum over an empty index set is defined to be 1. It is obvious from the form of the $L_{i,\ldots}$, that the degernaracy of the eigenvalues is in general removed if the l_i are mutually different. These functions are related to elementary symmetric functions in the variables l_i^2 where i runs in $\{1,\ldots,\mathrm{dim}V\}$ with $\{i,\ldots\}$ omitted in $L_{i,\ldots}$.

As special cases we notice, that for selfinverse metrics, $g=g^{-1}$ as matrices, we need to have $l_i=\pm 1$, and hence $l_i^2=1$. The eigenvalues are then given by the number of the terms in L_i times the Grassmann eigenvalues. This recovers Oziewicz's theorem that A is fully degenerate with eigenvalues $\mathrm{dim}V^\wedge=2^{\mathrm{dim}V}$. A second special case is $l_i=0$ for all i which reduces to the Grassmann case.

Let now $f: V\otimes V\to \Bbbk$ be a totally antisymmetric bilinear form and extend it as above via Laplace expansion to $f^\wedge$. As a consequence we see that the $\{u_i\}$ basis is no longer an eigenbasis to $A=m_f^{\mathrm{T}}\circ m_f$. The new eigenbasis introduces an f-dependent filtration of the algebra. This filtration can be turned into a gradation which was described by dotted wedge products in previous works (Fauser, 1996, 2001, 2002; Fauser and Abłamowicz, 2000). Exactly this new filtration establishes the Wick reordering of quantum field theory (Fauser, 2001). Hence a basis transformation in $V^\wedge$, acting as identity on V however, establishes the new gradation. A spectral decomposition of the product map has to use this new basis w.r.t. the newly established f-grading.

We know from cohomological considerations (Brouder *et al.*, 2003), that antisymmetric and symmetric twists fall into two classes, namely proper 2-cocycles and 2-coboundaries. This explains their different algebraic behaviors and allows to study the two cases independently. The general case is a convolutional mixture of these two possibilities. From group theory we know, that introducing a 2-cocycle might make it necessary to come up with the need of a change in the filtration of the algebra (Fauser and Jarvis, 2004).

Finally, we might remark, that the singular value decomposition allows to provide estimates on certain norms of the operators under consideration. The Frobenius norm of a $n \times m$ -map m is defined as

$$\sum_{i,j} m_{ij}^2 = \sum_k \left(d_k^{\frac{1}{2}}\right)^2 \tag{5.36}$$

while the operator 2-norm is given as

$$||m||_2 = \sup_{|v|=1} |mv| \; = \; d_1^{\frac{1}{2}} \tag{5.37}$$

where $d_1^{\frac{1}{2}}$ is the greatest singular value. In particular, we note that the Clifford and Grassmann multiplication maps are unbounded operators if $\dim V$ goes to infinity, e.g. is an L^2 space. The growth is exponential and the divergence hence serious.

5.3. Spectral form of Product Coproduct Pairs

Let m , $\Delta = m^{\mathrm{T}}$ be a product coproduct pair related by the Euclidean dual isomorphism, i.e. via transposition. Let $A = m \circ \Delta$ be the associated symmetric operator $A : V^\wedge \to V^\wedge$ with canonically normalized eigenvector basis $\{u_i\}$, $Au_i = \lambda_i u_i$. The $\{u_i\}$ form the column vectors of the matrix U which diagonalized A . Let $B = \Delta \circ m$, a symmetric operator, $B : V^\wedge \otimes V^\wedge \to V^\wedge \otimes V^\wedge$, having canonically normalized column eigenvectors $\{v_I\}$, which form the column vectors of the matrix V diagonalizing B . We can summarize our findings in the following

Theorem 5.8. *The coproduct $m^{\mathrm{T}} = \Delta$ maps the column eigenvectors u_i of A onto the column eigenvectors v_i of B w.r.t. the same singular value and vice versa the product maps the v_i onto the u_i . Let the canonical normalization be $UU^{\mathrm{T}} = D_A$ and $VV^{\mathrm{T}} = D_B$. Then the product has the spectral form*

$$m = \sum_i u_i \;\; \Delta(u_i)^{\mathrm{T}} \tag{5.38}$$

and the coproduct has the spectral form

$$\Delta = m^{\mathrm{T}} = \sum_{\{I\,|\,m(v_I)\neq 0\}} v_I \; m(v_I)^{\mathrm{T}} \,. \tag{5.39}$$

This amazing result technically allows to match the corresponding eigenvectors $\{u_i\}$ and $\{v_I\}$ via the Hopf algebra structure, since the coproduct exactly matches pairs of eigenvectors for a particular singular value. The computational technicallity of matching eigenvectores is hence resolved. Furthermore, the computation of the eigenvectors $\{u_i\}$ of A is considerably simpler than that computation of the eigenvectors $\{v_I\}$ for B, which can now be obtained from the application of the coproduct directly. The operators A and B are easily derived in spectral form as

$$\begin{aligned} A = m_g \circ m_g^{\mathrm{T}} &= \sum_{i,I} u_i \otimes (\Delta_g(u_i)^{\mathrm{T}} \mid v_I) \otimes m(v_I)^{\mathrm{T}} \\ B = m_g^{\mathrm{T}} \circ m &= \sum_{i,I} v_I \otimes (m(v_I)^{\mathrm{T}} \mid u_i) \otimes \Delta(u_i)^{\mathrm{T}} \end{aligned} \tag{5.40}$$

which holds true for *any* basis of A and $B = A \otimes A$ of course.

We mention here explicitely the technical importance of this result. As discussed in the introduction, SVD is a powerful and widely used tool for data compression, analysis of data, searching, image processing etc. A Hopf algebraic point of view, employing the computational accessible coproduct, may save lots of computation time and even bandwidth in transmitting data, since only the $\{u_i\}$ eigenvectores, and the singular values have to be sent, since the much more involved $\{v_I\}$ follow *uniquely* from the coproduct structure. Technical applications are based on the case studied in this paper, where product and coproduct are related by Euclidean duality, i.e. via transposition. In fortunate situations the coproduct may be known, and no information about it has to be transmitted at all. If the space A is graded, the information of the coproduct is reduced to the action on the grade 1 space and expanded using the homomorphism property Eq. (2.8). Of course, images may not have a product coproduct structures in general, so care is needed. However, see Abłamowicz (2002) for an embedding of matrix SVD into a Clifford algebraic setting.

6. CAS EXPERIMENT IN DIMENSION 2

Since its a difficult task to compute the singular values, vector space decompositions etc in the general twisted case, we consider here $\mathrm{dim} V = 2$ and use a computer algebra system (CAS) to solve the general setting for an arbitrary suitably chosen bilinear form B. We use CLIFFORD and BIGEBRA packages for Maple (2004).[6] Related results concerning the crossing, also derived using computer algebra can be found in Fauser and Oziewicz (2001).

Since we are mainly interested in a model which allows a physical interpretation, we choose the following nonsymmetric metric

$$B = \begin{bmatrix} 0 & \rho + \nu \\ \rho - \nu & 0 \end{bmatrix}. \tag{6.41}$$

The commutation and anticommutation relation for the e_i follow as

$$\begin{array}{c|cccc} \{\cdot \mid \cdot\}_+ & e_0 & e_1 & e_2 & e_{12} \\ \hline e_0 & 2\mathrm{Id} & 2e_1 & 2e_2 & 2e_{12} \\ e_1 & 2e_1 & 0 & 2\rho\mathrm{Id} & -2\nu e_1 \\ e_2 & 2e_2 & 2\rho\mathrm{Id} & 0 & -2\nu e_2 \\ e_{12} & 2e_{12} & -2\nu e_1 & -2\nu e_2 & 2(\rho^2 - \nu^2)\mathrm{Id} - 4\nu e_{12} \end{array} \tag{6.42}$$

$$\begin{array}{c|cccc} [\cdot \mid \cdot]_- & e_0 & e_1 & e_2 & e_{12} \\ \hline e_0 & 0 & 0 & 0 & 0 \\ e_1 & 0 & 0 & 2\nu\mathrm{Id} + 2e_{12} & -2\rho e_1 \\ e_2 & 0 & -2\nu\mathrm{Id} - 2e_{12} & 0 & 2\rho e_2 \\ e_{12} & 0 & 2\rho e_1 & -2\rho e_2 & 0 \end{array} \tag{6.43}$$

It is obvious that with the identification $a = e_1$, $a^\dagger = e_2$ we find that the canonical anticommutation relations (CAR)

$$\{a, a^\dagger\}_+ = 2\,\rho\mathrm{Id} \tag{6.44}$$

hold. For a detailed discussion of this and a 4-dimensional model see Fauser (2001a). The multiplication table is given as

$$m_B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \rho-\nu & 0 & 0 & \rho+\nu & 0 & 0 & 0 & 0 & 0 & \rho^2-\nu^2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & \rho-\nu & 0 & 0 & 0 & 0 & 0 & -\rho-\nu & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -\rho-\nu & 0 & 0 & \rho-\nu & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -2\nu \end{bmatrix} \tag{6.45}$$

The matrix $A = m \circ m^{\mathrm{T}}$ is hence given as

$$A = m_B \circ m_B^{\mathrm{T}} = \begin{bmatrix} a & 0 & 0 & b \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ b & 0 & 0 & d \end{bmatrix} \tag{6.46}$$

[6] A Maple worksheet containing the computations of this section is available from the author or from the url: http://clifford.physik.uni-konstanz.de/~fauser/.

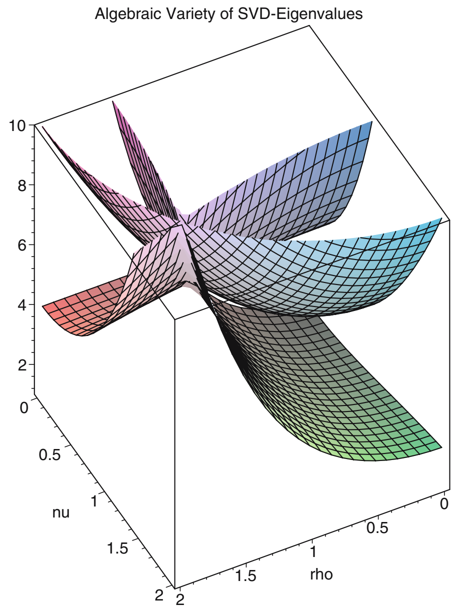

Fig. 1. Eigenvalue surfaces over the ρ - ν -plane. Shown is one quadrant, the other three are mirror symmetric w.r.t. the xz - and yz -planes. Remarkable is that all three planes meet in a one-dim. curve.

$$a = (\nu^2 + 1 + 2\rho\nu + \rho^2)(\nu^2 + 1 - 2\rho\nu + \rho^2) \quad b = 2\nu(1 - \rho^2 + \nu^2)$$
$$c = 2 + 2\rho^2 + 2\nu^2 \quad d = 4 + 4\nu^2 \tag{6.47}$$

We can identify the following special cases:

- $\rho = 0$ is the ν dependent Grassmann case. However, even in this case the deformed algebra obeys a new filtration, which is imposed by ν.
- A is diagonal, if $b = 0$, from which follows: $\rho = \pm\sqrt{1 + \nu^2}$ or $\nu = 0$. The eigenvalues are in this case of Clifford type and the fourfold degenerated eigenvalues are $4 + 4\nu^2$.

A remarkable fact is displayed in Fig. 1. All three[7] eigenvalue surfaces, emerging from the three types of rank, 0,1, and 2, meet in a single curve. This curve will be called singular locus, since it establishes a relation between ρ and ν in such a way that all eigenvalues are degenerated. In fact, this situation is

[7] Actually four surfaces, but two surfaces are degenerate, see Eq. (5.35).

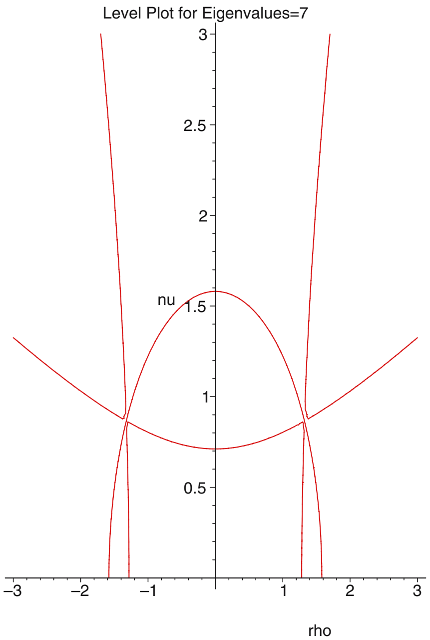

Fig. 2. Cross section for $z = 7$ of Fig. 1. The plot shows clearly the confocal level crossing of the eigenvalue surfaces.

singular in a much more peculiar way. The relation imposed, $\rho = \pm\sqrt{1 + \nu^2}$, also implies that the metric B on V squares (as a matrix) to one. Therefore, the coproduct is based on B^{-1} and the theorem of Oziewicz (see page 1742) stating that no antipode can exist applies in this case. In Fig. 1 we display the positive ρ - ν -quadrant of the algebraic varieties defined by the eigenvalues. The other quadrants are obtained by mirroring through the xz - and yz -planes. Two surfaces are saddle shaped, one has a (higher order) parabolic form. The incidence of all three surfaces is obvious from this plot.

In Fig. 2 we plot a section for constant z -value (i.e. $z = 7$). It is clearly seen how the surfaces intersect in a single curve (point in this section). Seen as eigenvalues, a *level-crossing* takes place, which is not correctly displayed in the plot, due to the contour plot option of Maple. One surface is doubly degenerated, since the matrix A has 4 eigenvalues, but due to the grading in our setting only three of them are different.

The commutation relations used in physics, having $\rho = \hbar/2$, does, in units of $\hbar$, *not* reach the degenerate case. This makes a difference only, if one assumes that a rescaling is not possible. Hence, if we agree that we have (half) integral

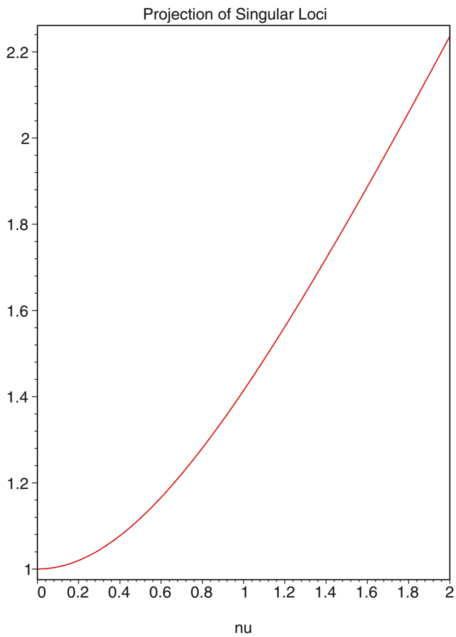

Fig. 3. Plot of the singular curve $\prod_{i\neq j}(L_i - L_j) = 0$, i.e. $\rho = \sqrt{1+\nu^2}$ of maximal degeneracy.

values for $\hbar$, measured in units of $\hbar$, we need to assume *higher spin values* to be realized to reach degeneracy. Since ν is not quantized, it can be arranged to hit the degeneracy, but only for sufficient large ρ. This correlation is displayed in Fig. 3 We plot there the projection of the singular curve into the ρ - ν -plane. Its easily seen that singularities need $\rho > 1$ to occur, that the asymptotics is $\rho(\nu) = 1$ for $\nu \to 0$ and $\rho(\nu) \simeq \nu + \text{const}$ for $\nu \to \infty$.

7. CONNECTION TO OTHER APPLICATIONS

Sine our proposed method to characterize products and coproducts via SVD is new, we do not give a conlusion, but a list of fields where SVD is used and our method might be applied. Some results were already obtained this way.

7.1. Symmetric Functions, Schur Functors

During the work on symmetric functions (Fauser and Jarvis, 2004) it became clear, that the homomorphism axiom, see Eq. (2.8), is equivalent to group branching rules. Our results on singular values suggest, that the split into degeneracy subspaces can be described by methods from invariant theory. In this sense, one can assume that the spaces are direct sums and carry a (quantum) group action. More over, the eigenvalues should then have a combinatorial interpretation and it should become possible to compute them in a more effective way. Hence looking in two different ways at the decomposition $U(n) \downarrow U(n) \otimes U(n)$ via product-coproduct or product-coproduct maps allows to connect the representations of the two sides also. Hence SVD is a Glebsch-Gordan problem in disguise. Knowing the branching rules is hence connected with knowing the spectral decomposition of the product and coproduct maps.

Classical invariant theory uses Schur functions to describe invariant subspaces. This method can be generalized to the functorial level where Schur functors characterize invariant subspaces as such, not supporting a basis. The main point is, that Schur functions allow, via the Littlewood-Richardson rule, the evaluation of the product $V_\lambda \otimes V_\mu = \sum_\nu c^\nu_{\lambda\mu} V_\nu$. In Fauser and Jarvis (2004) it was schown, how the cohomological Hopf algebra approch helpes to understand group branching laws. The SVD is hence connected to a direct computation of the invariant subspaces. This can be achieved by introducing new types of coproducts. E.g. we can pick involutions σ in V and define a new coproduct $\Delta_\sigma = (\sigma \otimes \sigma) \circ \Delta \circ \sigma$, which *cannot* be obtained via a deformation. Such a coproduct is able to produce elements in the kernel of m . In general, every transposition in S_n will allow to produce such a coproduct. These coproducts form in general no longer Hopf algebras together with the product under consideration. However, they are needed to construct algorithmically the kernel of the product map. One may consider

$$\Delta^-(e_i) = e_i \otimes \mathrm{Id} - \mathrm{Id} \otimes e_i \tag{7.48}$$

and extend it as a homomorphism, forcing a bialgebra structure

$$\Delta^-(m(A \otimes B)) = m(\Delta^-(A) \otimes \Delta^-(B)) \tag{7.49}$$

An example reads:

$$\begin{aligned}\Delta^-(e_1 \wedge e_2) &= (e_1 \otimes \mathrm{Id} - \mathrm{Id} \otimes e_1)(e_2 \otimes \mathrm{Id} - \mathrm{Id} \otimes e_2)\\ &= e_1 \wedge e_2 \otimes \mathrm{Id} - e_1 \otimes e_2 + e_2 \otimes e_1 + \mathrm{Id} \otimes e_1 \wedge e_2 \qquad (7.50)\\ m(\Delta^-(e_1 \wedge e_2) &= 0.\end{aligned}$$

Of course, its easy to see that $m(\Delta^-(A)) = 0$ and hence Δ^- has values in the kernel of m . Considering exact sequences as

$$0 \to \mathsf{Sym}_2(V^{\otimes^2}) \to V \otimes V \to V^{\wedge^2} \to 0 \tag{7.51}$$

shows then that the coproducts are involved in the construction of Schur functors, and Schur complexes, relating symmetric and antisymmetric powers of V . SVD will help to simplify and algorithmify this construction as will be demonstrated elsewhere, but see Akin *et al.* (1982).

7.2. Letter-Place Algebras, Invariant Theory

Gian-Carlo Rota developed the letter-place techniques to describe invariant theory on super algebras (Grosshans *et al.*, 1987). The Grassmann case treated in this work is the special case where all letters, i.e. formal variables, are negatively signed, hence anticommute. The pairing between two disjoint alphabets of letters, called letters and places, comes up with a neutral number, behaving like a scalar. Now, let letters be $L \cong L \otimes 1$ and places $P \cong 1 \otimes P$, a letter-place variable is given by the evaluation $[L \mid P] = \mathsf{eval}(L \otimes P) = L(P)$. The Homomorphism axiom in this case describes the evaluation and coevaluation of invariant theory. Therefore our above given treatment of SVD decompositions can be extended along this lines to graded or even braided linear algebra. The biorthogonality of a spectral decomposition should allow for more efficiency in super algebra algorithms.

7.3. Polar Decomposition of Operators

Another place, where SVD is used in disguise is the polar decomposition of operators. Let $A : W \to V$, $A^* : V \to W$, consider $A = \sqrt{A\,A^*}\,\frac{A}{\sqrt{A\,A^*}} = \rho\,\phi$. The operator ρ is a scaling operator, while ϕ is a 'phase'. In fact ρ^2 is our D_A and the inverse should be taken as generalized inverse, dropping the kernel. If we write $A = UD_A^{\frac{1}{2}}V^{\mathrm{T}}$, we get $\rho^2 = AA^* = UD_AU^{\mathrm{T}}$ and $\phi = UD_A^{-\frac{1}{2}}U^{\mathrm{T}}UD_A^{\frac{1}{2}}V^{\mathrm{T}} = UV^{\mathrm{T}}$ showing clearly that the scaling part goes into the ρ while the map UV^{T} describes the decomposition of the two tensor spaces W and V. This is related to our main theorem, which shows that Hopf algebras allow to compute $\Phi = UV^{\mathrm{T}} = U \circ \Delta(U)^{\mathrm{T}} = m(V) \circ V^{\mathrm{T}}$ using either the coproduct or the product map on the matrix column vectors. Looking at this decomposition in the SVD fashion allows to generalize it to singular and indefinite settings in a meaningful way. In fact, polar decompositions might be studied using branching laws too.

7.4. Numerical Applications

In numerical and computer applications, SVD is a well established method, a short discussion is found in Abłamowicz (2002). For applications in image processing, coding theory, noise reduction, latent semantic indexing, etc. see Maciejowski (1989), Strang (1998), Berry and Dongaara (1999).

7.5. Biorthogonalization in Biophysics

A further nice application of this seminal method is the so called 'Karhunen-Loewe' method, actually SVD, in chaos theory and in cerebral biology, see Kelso *et al.* (1992), Haken (1996), Bräuer (2002).

7.6. Manifold Theory—Function Valued Singular Values

We have skipped in the present work the complication that the duality in the eigenspaces of W and V may not be mediated by matrix transposition. We know from projective geometry and quantum field theory that coordinatizations can be done *independently* in point space and momentum space (of hyper planes or copoints) (Fauser and Stoss, 2004). This amounts to say, that we can pair two isomorphic but not identical Grassmann algebras $V^{\wedge}$ and V°_F}, where $\circ_F$ is another Grassmann product having a different filtration (F -grading) induced by the antisymmetric bilinear form $F^{\wedge}$. Such a freedom might be used to introduce a sort of 'metric' field into the branching scheme. As an example, one might think of morphisms which connect spaces only up to isomorphisms. Such a morphism would read in a basis

$$m_g : W \to V \quad m_g \cong \left[(m_g)^k_I\right] \tag{7.52}$$

where the indices are raised and lowered not by δ^K_I and δ^k_i but via an arbitrary, possibly function valued, $GL(V)$ element g_{ij} . Note that $g \otimes g \cong g_{IJ}$ is needed to raise/lower indices in W .

Having this generalization at our disposal, one might even think to put this as a bundle structure on a manifold, which then gives function valued metrics $g = g(x)$, x a basepoint of the manifold. We hope to investigate this elsewhere.

7.7. SVD and Cohomology

Cohomological considerations proved to be extremely useful in describing product structures in quantum field theory. The classification of such products and their explicite evaluation in a perturbative expansion was achieved using $\mathbb{C}$ -valued cohomology (Brouder *et al.*, 2003). However, if one consideres more complicated G -valued cohomology rings, or even cohomology monoids, the situation starts to get involved. Furthermore, cohomological methods are tied to topological invariants, hence are coarser that metric invariants. Having the SVD available, we can ask for metric invariants and the resulting eigenvalues carry metric information (due to the identifivation of V and V^*). We await therefore, that metrical information can be dealt with in the SVD approch better. This nourished the hope, expressed in the introduction, that we can unveil geometrical data of non-commutative manifolds this way.

ACKNOWLEDGEMENT

I would like to thank Peter Jarvis for many fruitful discussions about the properties of group branching laws and symmetric functions which influenced this work, and for a critical reading of the paper. Rafal Ablamowicz helped me

to understand the singular value decomposition in its Clifford algebra form. Kurt Bräuer led my attention to SVD applications in cerebral biology.

REFERENCES

Abe, E. (1980). *Hopf Algebras*. Cambridge University Press, Cambridge (transl. of the 1977 Japanese edition).

Abłamowicz, R. (2002). Hecke algebras, SVD and other computational examples with CLIFFORD. In *Applied Clifford Algebra in Cybernetics, Robotics, Image Processing and Engineering*, Garret Sobczyk and Eduardo Bayro-Corrochano, eds., ACACSE:99. Birkh 2002. extended version from math.RA/9910069.

Abłamowicz, R., and Fauser, B. (1996–2002). *CLIFFORD*. Tennessee Technological University, http://math.tntech.edu/rafal/, Maple V program package.

Abłamowicz, R., and Fauser, B. (1999–2002). *BIGEBRA*. Tennessee Technological University and University of Konstanz, http://math.tntech.edu/rafal/, Maple V program package.

Akin, K., Buchsbaum, D. A., and Weyman, J. (1982). Schur unctors and Schur complexes. *Advances in Mathematics* **44**, 207–278.

Berry, M., and Dongaara, J. (1999). Atlanta organizers put mathematics to work for the math sciences community. *SIAM News* **32**(6), and references therein.

Bräuer, K. (2002). *Chaos, Attraktoren und Fraktale: mathematische und physikalische Grundlagen nichtlinearer Phänomene mit Anwendungen in Physik, Biologie und Medizin*, Logos-Verlag, Berlin.

Brouder, C., Fauser, B., Frabetti, A., and Oeckl., R. (2003). Let's twist again. preprint [hep-th/0311253].

Brouder, C., and Schmitt, W. (2002). Quantum groups and quantum field theory III. Renormalization. preprint, pp. 1–18 [hep-th/0210097].

Chryssomalakos. C. (1998). Quantum and braided integrals. In *JHEP Proceedings of the Corfu Summer Institute on Elementary Particle Physics*, Corfu, Greece [math-ph/0104006].

Fauser, B. (1996). *Clifford-algebraische Formulierung und Regularität in der Quantenfeldtheorie*. PhD thesis, Universität Tübingen, Tübingen.

Fauser, B. (2001). Clifford geometric parameterization of inequivalent vacua. *Mathematical Methods in the Applied Sciences* **24**, 885–912 [hep-th/9710047v2].

Fauser, B. (2001). On the Hopf-algebraic origin of Wick normal-ordering. *Journal of Physics A: Mathematical and General* **34**, 105–115 [hep-th/0007032].

Fauser, B. (2002). A Treatise on Quantum Clifford Algebras. Konstanz, Habilitationsschrift, [arXiv:math.QA/0202059].

Fauser, B., and Abłamowicz, R. (2000). On the decomposition of Clifford algebras of arbitrary bilinear form. In *Clifford Algebras and their Applications in Mathematical Physics*, Boston, Birkhäuser, pp. 341–366 [math.QA/9911180].

Fauser, B., and Jarvis, P. D. (2004). A Hopf laboratory for symmetric functions. *Journal of Physics A: Mathematical and General* **37**(5), 1633–1663 [math-ph/0308043].

Fauser, B., and Oziewicz, Z. (2001). Clifford Hopf gebra for two dimensional space. *Miscellanea Algebraicae* **2**(1), 31–42 [math.QA/0011263].

Fauser, B., and Stoß, H.-J. (2004). Consequences of independent coordinatizations for points and hyperplanes. preprint.

Grosshans, F. D., Rota, G. C., and Stein, J. A. (1987). Invariant theory and superalgebras. In *Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics*, no. 69, American Mathematical Society, pp. i–xxi, 1–80.

Hahn, A. J. (1994). *Quadratic Algebras, Clifford Algebras, and Arithmetic Witt Groups*. Springer, Berlin; Heidelberg.

Haken, H. (1996). *Principles of Brain Functioning: A Synergetic Approach to Brain Activity, Behavior and Cognition*. Springer, Berlin; Heidelberg; New York, Springer series in synergetics; 67.

Hirshfeld, A. C., and Henselder, P. (2002). Star products and perturbative quantum field theory. *Annales Physics* **298**, 382–393 [hep-th/0208194].

Kassel, C. (1995). *Quantum Groups*. Springer-Verlag, New-York, Berlin: Heidelberg.

Kelso, J. A. S., Bressler, S. L., Buchanan, S., DeGuncman, G. C., Ding, M., Fuchs, A., and Holroyd, T. (1992). A phase transition in human brain and behaviour. *Physics Letters A* **169**, 134–144.

Maciejowski, J. M. (1989). *Multivariable Feedback Design*. Addison-Wesley, Wokingham, England.

Majid, S. (1995). *Foundations of Quantum Group Theory*. Cambridge University Press, Cambridge.

Maple, W. (2004). *Maple Release V to 9*. Maplesoft, University of Walterloo, CA [http://www.maplesoft.com].

Oziewicz, Z. (1997). Clifford Hopf gebra and biuniversal Hopf gebra. *Czechoslovak Journal of Physics* **47**(12), 1267–1274 [q-alg/9709016].

Oziewicz, Z. (2001). Guest editor's note: Clifford algebras and their applications. *International Journal of Theoretical Physics* **40**(1), 1–13.

Oziewicz, Z. (2003). Operad of graphs, convolution and Hopf gebra. *Contemporary Mathematics* **318**, 175–197.

Rota, G.-C., and Stein, J. A. (1994). Plethystic Hopf algebras. *Proceedings of the National Academy of Sciences of the United States of America* **91**, 13057–13061.

Strang, G. (1998). *Introduction to Linear Algebra*. Wellesley-Cambridge Press, Wellesley.

Sweddler, M. E. (1968). Cohomology of algebras over Hopf algebras. *Transactions of the American Mathematical Society* **133**, 205–239.

Sweedler, M. E. (1969). *Hopf Algebras*. W. A. Benjamin, INC., New York.